Pontryagin duality for Abelian s- and sb-groups☆

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Abstract

The main goal of the article is to study the Pontryagin duality for Abelian s- and sb-groups. Let G be an infinite Abelian group and X be the dual group of the discrete group G_d. We show that a dense subgroup H of X is $\mathfrak{g}$-closed iff H algebraically is the dual group of G endowed with some maximally almost periodic s-topology. Every reflexive Polish Abelian group is $\mathfrak{g}$-closed in its Bohr compactification. If a s-topology τ on a countably infinite Abelian group G is generated by a countable set of convergent sequences, then the dual group of (G,τ) is Polish. A non-trivial Hausdorff Abelian topological group is a s-group iff it is a quotient group of the s-sum of a family of copies of $(\mathbb{Z}_0^{\mathbb{N}}, \mathbf{e})$.

Keywords:
T-sequence, TB-sequence, Abelian group, s-group, sb-group, dual group, $\mathfrak{g}$-closed subgroup, sequentially covering map
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1. Introduction

I. Notations and preliminaries result. A group G with the discrete topology is denoted by G_d. The subgroup generated by a subset A of G is denoted by $\langle A\rangle$. Let X be an Abelian topological group. A basis of open neighborhoods at zero of X is denoted by $\mathcal{U}_X$. The group of all continuous characters on X is denoted by $\widehat{X}$. $\widehat{X}$ endowed with the compact-open

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topology is denoted by $X^\wedge$. Denote by $\mathbf{n}(X) = \cap_{\chi\in\widehat{X}}\ker\chi$ the von Neumann radical of X. If $\mathbf{n}(X) = \{0\}$, X is called maximally almost periodic (MAP).

Let X be an Abelian topological group and $\mathbf{u} = \{u_n\}$ be a sequence of elements of $\widehat{X}$. Following Dikranjan et al. [7], we denote by $s_{\mathbf{u}}(X)$ the set of all $x \in X$ such that $(u_n, x) \to 1$. Let H be a subgroup of X. If $H = s_{\mathbf{u}}(X)$ we say that $\mathbf{u}$ *characterizes* H and that H is *characterized* (by $\mathbf{u}$) [7]. Let X be a metrizable compact Abelian group. By [10, Corollary 1], each characterized subgroup $H = s_{\mathbf{u}}(X)$ admits a locally quasi-convex Polish group topology. We denote the group $H = s_{\mathbf{u}}(X)$ with this topology by $H_{\mathbf{u}}$.

Let H be a subgroup of an Abelian topological group X. Following [7], the closure operator $\mathfrak{g}_X$ is defined as follows

$$\mathfrak{g}(H) = \mathfrak{g}_X(H) := \bigcap_{\mathbf{u}\in\widehat{X}^{\mathbb{N}}} \{s_{\mathbf{u}}(X) : \ H \leq s_{\mathbf{u}}(X)\},$$

and we say that H is $\mathfrak{g}$-*closed* if $H = \mathfrak{g}(H)$. For an arbitrary subset S of $\widehat{X}^{\mathbb{N}}$, one puts

$$s_S(X) := \bigcap_{\mathbf{u}\in S} s_{\mathbf{u}}(X).$$

Let $\mathbf{u} = \{u_n\}$ be a non-trivial sequence in an Abelian group G. The following very important questions has been studied by many authors as Graev [13], Nienhuys [16], and others:

Problem 1.1. *Is there a Hausdorff group topology τ on G such that $u_n \to 0$ in (G, τ)?*

Protasov and Zelenyuk [19, 20] obtained a criterion that gives the complete answer to this question. Following [19], we say that a sequence $\mathbf{u} = \{u_n\}$ in a group G is a T-*sequence* if there is a Hausdorff group topology on G in which u_n converges to 0. The group G equipped with the finest Hausdorff group topology $\tau_{\mathbf{u}}$ with this property is denoted by $(G, \mathbf{u})$. A T-sequence $\mathbf{u} = \{u_n\}$ is called trivial if there is n_0 such that $u_n = 0$ for every $n \geq n_0$.

Let G be a countably infinite Abelian group, $X = G_d^\wedge$, $\mathbf{u} = \{u_n\}$ be a T-sequence in G and $H = s_{\mathbf{u}}(X)$. There is a simple dual connection between the groups $(G, \mathbf{u})$ and $H_{\mathbf{u}}$, and, moreover, we can compute the von Neumann radical $\mathbf{n}(G, \mathbf{u})$ of $(G, \mathbf{u})$ as follows:

Theorem 1.2. [10] *$(G, \mathbf{u})^\wedge = H_{\mathbf{u}}$ and, algebraically, $\mathbf{n}(G, \mathbf{u}) = H^\perp$.*

The counterpart of Problem 1.1 for *precompact* group topologies on $\mathbb{Z}$ is studied by Raczkowski [17]. Following [2] and motivated by [17], we say that a sequence $\mathbf{u} = \{u_n\}$ is a *TB-sequence* in an Abelian group G if there is a *precompact* Hausdorff group topology on G in which $u_n \to 0$. The group G equipped with the finest precompact Hausdorff group topology $\tau_{b\mathbf{u}}$ with this property is denoted by $(G, b\mathbf{u})$.

For an Abelian group G and an arbitrary subgroup $H \leq G_d^\wedge$, let T_H be the weakest topology on G such that all characters of H are continuous with respect to T_H. One can easily show [5] that T_H is a totally bounded group topology on G, and it is Hausdorff iff H is dense in $G_d^\wedge$.

A subset A of a topological space Ω is called *sequentially open* if whenever a sequence $\{u_n\}$ converges to a point of A, then all but finitely many of the members u_n are contained in A. The space Ω is called *sequential* if any subset A is open if and only if A is sequentially open. Franklin [9] gave the following characterization of sequential spaces:

Theorem 1.3. [9] *A topological space is sequential if and only if it is a quotient of a metric space.*

The following natural generalization of Problem 1.1 was considered in [12]:

Problem 1.4. *Let G be a group and S be a set of sequences in G. Is there a (resp. precompact) Hausdorff group topology τ on G in which every sequence of S converges to zero?*

By analogy with T- and TB-sequences, we define [12]:

Definition 1.5. *Let G be an Abelian group and S be a set of sequences in G. The set S is called a TS-set (resp. TBS-set) of sequences if there is a Hausdorff (resp. precompact Hausdorff) group topology on G in which all sequences of S converge to zero. The finest Hausdorff (resp. precompact Hausdorff) group topology with this property is denoted by τ_S (resp. τ_{bS}).*

The set of all TS-sets (resp. TBS-sets) of sequences of a group G we denote by $\mathcal{TS}(G)$ (resp. $\mathcal{TBS}(G)$). It is clear that, if $S \in \mathcal{TS}(G)$ (resp. $S \in \mathcal{TBS}(G)$), then $S' \in \mathcal{TS}(G)$ (resp. $S' \in \mathcal{TBS}(G)$) for every nonempty subset S' of S and every sequence $\mathbf{u} \in S$ is a T-sequence (resp. $\mathbf{u}$ is a TB-sequence). Evidently, $\tau_S \subseteq \tau_{S'}$ (resp. $\tau_{bS} \subseteq \tau_{bS'}$). Also, if S contains only trivial T-sequences, then $S \in \mathcal{ST}(G)$ and τ_S is discrete.

By definition, $\tau_{\mathbf{u}}$ is finer than τ_S (resp. $\tau_{b\mathbf{u}}$ is finer than τ_{bS}) for every $\mathbf{u} \in S$. Thus, if U is open (resp. closed) in τ_S, then it is open (resp. closed) in $\tau_{\mathbf{u}}$ for every $\mathbf{u} \in S$. So, by definition, we obtain that $\tau_S \subseteq \bigwedge_{\mathbf{u}\in S} \tau_{\mathbf{u}}$ (resp. $\tau_{bS} \subseteq \bigwedge_{\mathbf{u}\in S} \tau_{b\mathbf{u}}$).

The following class of topological groups is defined in [12]:

Definition 1.6. *A Hausdorff Abelian topological group (G, τ) is called a s-group (resp. a bs-group) and the topology τ is called a s-topology (resp. a bs-topology) on G if there is $S \in \mathcal{TS}(G)$ (resp. $S \in \mathcal{TBS}(G)$) such that $\tau = \tau_S$ (resp. $\tau = \tau_{bS}$).*

In other words, s-groups are those topological groups whose topology can be described by a set of convergent sequences. The family of all Abelian s-group is denoted by **SA**.

One of the most natural way how to find TS-sets of sequences is as follows. Let (G, τ) be a Hausdorff Abelian topological group. We denote the set of all sequences of (G, τ) converging to zero by $S(G, \tau)$:

$$S(G, \tau) = \{\mathbf{u} = \{u_n\} \subset G : \ u_n \to 0 \text{ in } \tau\}.$$

It is clear, that $S(G, \tau) \in \mathcal{TS}(G)$ and $\tau \subseteq \tau_{S(G,\tau)}$. The group $\mathbf{s}(G, \tau) := (G, \tau_{S(G,\tau)})$ is called the s-refinement of (G, τ) [12].

In [12] it is proved that the class **SA** is closed under taking of quotient and it is finitely multiplicative. It is natural that this class contains all sequential groups [12]. For every countable TS-set of sequences in an Abelian group G the space (G, τ_S) is complete and sequential (see [12]). Another non-trivial examples of sequential Hausdorff Abelian groups see [4]. A complete description of Abelian s-groups is given in [12].

Let X and Y be topological groups. Following Siwiec [18], a continuous homomorphism $p : X \to Y$ is called *sequence-covering* if and only if it is surjective and for every sequence $\{y_n\}$ converging to the unit e_Y there is a sequence $\{x_n\}$ converging to e_X such that $p(x_n) = y_n$.

II. Main results. The main goal of the article is to study the Pontryagin duality for Abelian s- and sb-groups. We give a simple dual connection between the MAP s-topologies on an infinite Abelian group G and the dense $\mathfrak{g}$-closed subgroups of the compact group $G_d^\wedge$. Also we describe all bs-topologies on G.

The article is organized as follows. In Section 2 we study the dual groups of Abelian s- and sb-groups and prove the following generalization of the algebraic part of Theorem 1.2:

Theorem 1.7. *Let $S \in \mathcal{TS}(G)$ for an infinite Abelian group G and $i_S : G_d \to (G, \tau_S), i_S(g) = g$, be the natural continuous isomorphism. Then*

1) $i_S^\wedge\left((G, \tau_S)^\wedge\right) = s_S(G_d^\wedge)$;
2) $\mathbf{n}(G, \tau_S) = [s_S(G_d^\wedge)]^\perp$ *algebraically.*

Also, in this section, we describe all sb-topologies on an infinite Abelian group G. In [7], it was pointed out that $\tau_{b\mathbf{u}} = T_{s_{\mathbf{u}}(G_d^\wedge)}$ for one TB-sequence $\mathbf{u}$. The following theorem generalizes this fact:

Theorem 1.8. *Let $S \in \mathcal{TBS}(G)$ for an infinite Abelian group G and $j_S : G_d \to (G, \tau_{bS}), j_S(g) = g$, be the natural continuous isomorphism. Then*

1) $j_S^\wedge\left((G, \tau_S)^\wedge\right) = s_S(G_d^\wedge)$;
2) $\tau_{bS} = T_{s_S(G_d^\wedge)}$.

As an immediate corollary of Theorems 1.7 and 1.8 we obtain:

Corollary 1.9. *Let $S \in \mathcal{TBS}(G)$ for an infinite Abelian group G and $j : (G, \tau_S) \to (G, \tau_{bS}), j(g) = g$, be the natural continuous isomorphism. Then its conjugate homomorphism $j^\wedge : (G, \tau_{bS})^\wedge \to (G, \tau_S)^\wedge$ is a continuous isomorphism.*

Using Theorem 1.7 we obtain the following dual connection between dual groups of s-groups and $\mathfrak{g}$-closed subgroups of compact Abelian groups:

Theorem 1.10. *Let G be an infinite Abelian group. Set $X = G_d^\wedge$.*

(i) *If $S \in \mathcal{TS}(G)$ and $i_S : G_d \to (G, \tau_S)$ is the natural continuous isomorphism, then $i_S^\wedge\left((G, \tau_S)^\wedge\right)$ is a $\mathfrak{g}$-closed subgroup of X.*
(ii) *If H is a $\mathfrak{g}$-closed subgroup of X, then there is $S \in \mathcal{TBS}\left((\mathrm{cl}H)^\wedge\right)$ such that $H = \left((\mathrm{cl}H)^\wedge, \tau_S\right)^\wedge$ algebraically.*

The following two statements are immediately corollaries of Theorem 1.10 and the fact that every sequential group is a s-group [12]:

Corollary 1.11. *A dense subgroup H of an infinite compact Abelian group X is $\mathfrak{g}$-closed if and only if H algebraically is the dual group of $\widehat{X}$ endowed with some MAP s-topology.*

Corollary 1.12. *The dual group of a sequential group (G, τ) is a $\mathfrak{g}$-closed subgroup of the compact group $G_d^\wedge$.*

For an Abelian topological group (G, τ), $bG = \widehat{G}_d^\wedge$ denotes its Bohr compactification. We shall identify G, if it is MAP, and $G^{\wedge\wedge}$ with their images in bG (see below Section 2). From Corollary 1.11 we obtain:

Corollary 1.13. *Let an Abelian topological group (G, τ) be such that $G^\wedge$ is a s-group. Then $G^{\wedge\wedge}$ is a dense $\mathfrak{g}$-closed subgroup of bG.*

Further, we show:

Proposition 1.14. *Every reflexive Polish Abelian group (in particular, every separable locally convex Banach space or separable metrizable locally compact Abelian group) is $\mathfrak{g}$-closed in its Bohr compactification.*

In [12], a general criterion to be a s-group is given. In Section 3 we obtain another analog of Franklin's theorem 1.3 for Abelian s-groups. Let $\{G_i\}_{i\in I}$, where I is a non-empty set of indices, be a family of Abelian groups. The direct sum of G_i is denoted by

$$\sum_{i\in I} G_i := \left\{ (g_i)_{i\in I} \in \prod_{i\in I} G_i : \ g_i = 0 \text{ for almost all } i \right\}.$$

We denote by j_k the natural including of G_k into $\sum_{i\in I} G_i$, i.e.:

$$j_k(g) = (g_i) \in \sum_{i\in I} G_i, \text{ where } g_i = g \text{ if } i = k \text{ and } g_i = 0 \text{ if } i \neq k.$$

Let $G_i = (G_i, \tau_i)$ be an Abelian s-group for every $i \in I$. It is easy to show that the set $\bigcup_{i\in I} j_i\left(S(G_i, \tau_i)\right)$ is a TS-set of sequences in $\sum_{i\in I} G_i$ (see Section 4).

Definition 1.15. *Let $\{(G_i, \tau_i)\}_{i\in I}$ be a non-empty family of Abelian s-groups. The group $\sum_{i\in I} G_i$ endowed with the finest Hausdorff group topology τ^s in which every sequence of $\bigcup_{i\in I} j_i\left(S(G_i, \tau_i)\right)$ converges to zero is called the* **s-sum** *of G_i and it is denoted by $s - \sum_{i\in I} G_i$.*

By definition, the s-sum of s-groups is a s-group either. Note that the s-sum of s-groups can be defined also for non-Abelian s-groups.

Set $\mathbb{Z}_0^{\mathbb{N}} = \{(n_1, \dots, n_k, 0, \dots) | n_j \in \mathbb{Z}\}$ and $\mathbf{e} = \{e_n\} \in \mathbb{Z}_0^{\mathbb{N}}$, where $e_1 = (1, 0, 0, \dots), e_2 = (0, 1, 0, \dots), \dots$. Then $\mathbf{e}$ is a T-sequence in $\mathbb{Z}_0^{\mathbb{N}}$. The following theorem gives a characterization of Abelian s-groups and it can be considered as a natural analog of Franklin's theorem 1.3:

Theorem 1.16. *Let (X, τ) be a non-discrete Hausdorff Abelian topological group. The following statements are equivalent:*

(i) *(X, τ) is a s-group;*

(ii) *(X, τ) is a quotient group of the s-sum of a non-empty family of copies of $(\mathbb{Z}_0^{\mathbb{N}}, \mathbf{e})$. Moreover, a quotient map may be chosen to be sequence-covering.*

Let G be an infinite Abelian group. In Section 4 we consider the case of countable $S \in \mathcal{TS}(G)$. In this case, the topology τ_S has a simple description (see Proposition 2.2). The main result of the section is the following:

Theorem 1.17. *Let G be a countably infinite Abelian group and let $S = \{\mathbf{u}_n\}_{n\in\omega} \in \mathcal{TS}(G)$. Then $(G, \tau_S)^\wedge$ is a Polish group. More precisely, $(G, \tau_S)^\wedge$ embeds onto a closed subgroup of the Polish group $\prod_{n\in\omega}(G, \mathbf{u}_n)^\wedge$.*

As a corollary we prove the following two propositions (see Problems 2.21 and 2.22 [11]):

Proposition 1.18. *Let $\{X_n\}_{n\in\omega}$ be a sequence of second countable locally compact Abelian groups. Then there is a complete countably infinite Abelian MAP s-group (G, τ) such that*

$$(G, \tau)^\wedge = \prod_{n\in\omega} X_n.$$

Proposition 1.19. *There is a complete sequential MAP group topology τ on $\mathbb{Z}_0^{\mathbb{N}}$ such that*

$$(\mathbb{Z}_0^{\mathbb{N}}, \tau)^\wedge = \mathbb{R}^{\mathbb{N}}.$$

In the last section we pose some open questions.

2. Duality

The following lemma will be used several times in the article:

Lemma 2.1. [7, Lemma 3.1] *Let G be an Abelian topological group and let $H \le G^\wedge$. Then, for a sequence $\mathbf{u} = \{u_n\}$ in G, one has $u_n \to 0$ in (G, T_H) if and only if $H \le s_{\mathbf{u}}(G^\wedge)$.*

The following proposition connects the notions of T- and TB-sequences (for countably infinite G see [10]).

Proposition 2.2. *Let $\mathbf{u} = \{u_n\}$ be a sequence in an Abelian group G. Then*

(i) $\mathbf{u}$ *is a TB-sequence if and only if it is a T-sequence and $(G, \mathbf{u})$ is MAP.*

(ii) *Let $\mathbf{u}$ be a TB-sequence and let $i_{\mathbf{u}} : G_d \to (G, \mathbf{u})$ and $j_{\mathbf{u}} : G_d \to (G, b\mathbf{u})$, $i_{\mathbf{u}}(g) = j_{\mathbf{u}}(g) = g$, be the natural continuous isomorphisms. Then $i^{\wedge}_{\mathbf{u}}((G, \mathbf{u})^{\wedge}) = j^{\wedge}_{\mathbf{u}}((G, b\mathbf{u})^{\wedge}) = s_{\mathbf{u}}(G^{\wedge}_d)$.*

(iii) [7] *If $\mathbf{u}$ is a TB-sequence, then $\tau_{b\mathbf{u}} = T_{s_{\mathbf{u}}(G^{\wedge}_d)}$.*

Proof. Set $H = i^{\wedge}_{\mathbf{u}}((G, \mathbf{u})^{\wedge})$, $Y = j^{\wedge}_{\mathbf{u}}((G, b\mathbf{u})^{\wedge})$ and $h : (G, \mathbf{u}) \to (G, b\mathbf{u})$, $h(g) = g$. Then h is a continuous isomorphism and $j_{\mathbf{u}} = h \circ i_{\mathbf{u}}$. So $j^{\wedge}_{\mathbf{u}} = i^{\wedge}_{\mathbf{u}} \circ h^{\wedge}$ and $Y \subseteq H$.

(i) It is clear that, if a sequence $\mathbf{u} = \{u_n\}$ is a TB-sequence, then it is a T-sequence and $(G, \mathbf{u})$ is MAP. Let us prove the converse assertion. Let $x \in H$ and $x = i^{\wedge}_{\mathbf{u}}(\chi), \chi \in (G, \mathbf{u})^{\wedge}$. Then $(u_n, x) = (i_{\mathbf{u}}(u_n), \chi) \to 1$. Thus $Y \subseteq H \subseteq s_{\mathbf{u}}(G^{\wedge}_d)$. Hence, by Lemma 2.1, $u_n \to 0$ in T_H. Since $i^{\wedge}_{\mathbf{u}}$ is injective and $(G, \mathbf{u})$ is MAP, the topology T_H is Hausdorff. Since T_H is precompact, $\mathbf{u}$ is a TB-sequence.

(ii) We claim that $Y = H = s_{\mathbf{u}}(G^{\wedge}_d)$. Indeed, by Lemma 2.1, $u_n \to 0$ in $T_{s_{\mathbf{u}}(G^{\wedge}_d)}$. Thus the topology $\tau_{b\mathbf{u}}$ is finer than $T_{s_{\mathbf{u}}(G^{\wedge}_d)}$. Hence, by [5, 1.2 and 1.4], we have $s_{\mathbf{u}}(G^{\wedge}_d) \subseteq Y$. By item (i) of the proof, we obtain that $Y = H = s_{\mathbf{u}}(G^{\wedge}_d)$.

(iii) follows from item (ii) and [5, Theorem 1.2].

Proof of Theorem 1.7. 1) By definition, the natural inclusions $i_{\mathbf{u}} : G_d \to (G, \mathbf{u})$ and $t_{\mathbf{u}} : (G, \mathbf{u}) \to (G, \tau_S)$, $i_{\mathbf{u}}(g) = t_{\mathbf{u}}(g) = g$, are continuous isomorphisms for every $\mathbf{u} \in S$, and $i_S = t_{\mathbf{u}} \circ i_{\mathbf{u}}$. By Proposition 2.2(ii), $i^{\wedge}_{\mathbf{u}}((G, \mathbf{u})^{\wedge}) = s_{\mathbf{u}}(G^{\wedge}_d)$. Hence $i^{\wedge}_S((G, \tau_S)^{\wedge}) \subseteq s_{\mathbf{u}}(G^{\wedge}_d)$ for every $\mathbf{u} \in S$. So $i^{\wedge}_S((G, \tau_S)^{\wedge}) \subseteq s_S(G^{\wedge}_d)$.

Conversely, let $x \in s_S(G^{\wedge}_d)$. By Proposition 2.2(ii), $x \in i^{\wedge}_{\mathbf{u}}((G, \mathbf{u})^{\wedge})$ for every $\mathbf{u} = \{u_n\} \in S$. Thus, x is an algebraic homomorphism from (G, τ_S) into $\mathbb{T}$ such that, by the definition of the topology $\tau_{\mathbf{u}}$, $(u_n, x) \to 1$ for every $\mathbf{u} \in S$. By [12, Theorem 2.4], x is a continuous character of (G, τ_S). So $x \in i^{\wedge}_S((G, \tau_S)^{\wedge})$.

2) By 1), algebraically we have

$$\mathbf{n}(G, \tau_S) = \bigcap_{\chi \in (G, \tau_S)^{\wedge}} \ker \chi = \bigcap_{x \in s_S(G^{\wedge}_d)} \ker x = \left[s_S(G^{\wedge}_d)\right]^{\perp}. \qquad \square$$

Corollary 2.3. *Let G be an infinite Abelian group and S be an arbitrary set of sequences in G. Then the following statements are equivalent:*

1) *$S \in \mathcal{TBS}(G)$;*
2) *$S \in \mathcal{TS}(G)$ and (G, τ_S) is MAP;*
3) *$s_S(G_d^\wedge)$ is dense in $G_d^\wedge$.*

PROOF. 1) $\Rightarrow$ 2) is trivial.

2) $\Rightarrow$ 3) By Theorem 1.7, (G, τ_S) is MAP iff $s_S(G_d^\wedge)$ is dense in $G_d^\wedge$.

3) $\Rightarrow$ 1) For simplicity we set $H := s_S(G_d^\wedge)$. By Lemma 2.1, every $\mathbf{u} \in S$ converges to zero in T_H. Since H is dense in $G_d^\wedge$, by [5, Theorem 1.9], T_H is Hausdorff. So $S \in \mathcal{TBS}(G)$.

Proof of Theorem 1.8. Set $H := s_S(G_d^\wedge)$ and $Y = j_S^\wedge((G, \tau_{bS})^\wedge)$. By Theorem 1.7, $H = i_S^\wedge((G, \tau_S)^\wedge)$. Since τ_S is finer than τ_{bS}, the identity map $j : (G, \tau_S) \to (G, \tau_{bS})$ is a continuous isomorphism. Hence $j^\wedge : (G, \tau_{bS})^\wedge \to (G, \tau_S)^\wedge$ is a continuous monomorphism. Since $j_S^\wedge = i_S^\wedge \circ j^\wedge$, we have $Y \subseteq H$. By Lemma 2.1, every $\mathbf{u} \in S$ converges to zero in T_H. Thus, τ_{bS} is finer than T_H. Hence, by [5, 1.2 and 1.4], $H \subseteq Y$. Thus, $H = Y$. By [5, Theorems 1.2 and 1.3], $\tau_{bS} = T_H$. $\Box$

The following corollary generalizes [7, Proposition 3.2].

Corollary 2.4. *Let G be an infinite Abelian group and $S \in \mathcal{TBS}(G)$. Then:*

1) *$\omega(G, \tau_{bS}) = |s_S(G_d^\wedge)|$;*
2) *τ_{bS} is metrizable iff $s_S(G_d^\wedge)$ is countable.*

PROOF. 1) follows from Theorem 1.8 and the property $\omega(G, T_Y) = |Y|$ of the topology T_Y generated by any subgroup $Y \leq G_d^\wedge$.

Put $H := s_S(G_d^\wedge)$. By Corollary 2.3, H is a point-separating subgroup of $G_d^\wedge$. Thus 2) follows from Theorem 1.8 and [5, Theorem 1.11].

Remark 1. Note that, in general, for τ_S the equality $\omega(G, \tau_S) = |s_S(G_d^\wedge)|$ is not fulfilled. Indeed, let $S = \{\mathbf{u}\}$ contain only one T-sequence and let S be such that $s_S(G_d^\wedge)$ is countably infinite and dense in $G_d^\wedge$ [6]. Then $|s_S(G_d^\wedge)| = \aleph_0$, but (G, τ_S) is not metrizable [19], and hence $\omega(G, \tau_S) > \aleph_0$. Now, let (G, τ) be a dense countable subgroup of a locally compact non-compact Abelian metrizable group X with the induced topology and $S = S(G, \tau)$. By [12, Theorem 1.13], $\omega(G, \tau_S) = \omega(G, \tau) = \aleph_0$, but $|s_S(G_d^\wedge)| = |X^\wedge| = \mathfrak{c}$.

As usual, the natural homomorphism from an Abelian topological group G into its bidual group $G^{\wedge\wedge}$ is denoted by α.

Corollary 2.5. *Let G be a countably infinite Abelian group and $S \in \mathcal{TBS}(G)$. If $s_S(G_d^\wedge)$ is countable, then $(G, \tau_{bS})^\wedge$ is discrete. In particular, (G, τ_{bS}) is not reflexive.*

PROOF. By Corollary 2.4, τ_{bS} is metrizable. Hence the completion $\overline{G}$ of (G, τ_{bS}) is a metrizable compact group. Thus, $\overline{G}$ is determined [1, 3]. Hence $(G, \tau_{bS})^\wedge$ is topologically isomorphic to the discrete group $\overline{G}^\wedge$. In particular, $(G, \tau_{bS})^{\wedge\wedge}$ is an infinite compact group. Since G is countable, $\alpha(G) \neq (G, \tau_{bS})^{\wedge\wedge}$. So (G, τ_{bS}) is not reflexive.

Corollary 2.6. *Let G be a countably infinite Abelian group, $\mathbf{u}$ be a TB-sequence and $j : (G, \mathbf{u}) \to (G, b\mathbf{u})$ be the identity continuous isomorphism. If $s_{\mathbf{u}}(G_d^\wedge)$ is countable, then $j^\wedge$ is a topological isomorphism.*

PROOF. $j^\wedge$ is a continuous isomorphism by Corollary 1.9. Since, by Theorem 1.2, $(G, \mathbf{u})^\wedge$ is complete and countable, it is discrete. Thus $(G, \tau_{bS})^\wedge$ is also discrete and $j^\wedge$ is a topological isomorphism. □

Proof of Theorem 1.10. (i) follows from Theorem 1.7(1) and the definitions of $\mathfrak{g}$-closed subgroups.

(ii) It is clear that H is a $\mathfrak{g}$-closed dense subgroup of $\mathrm{cl}H$. Put

$$S := \{\mathbf{u} \in ((\mathrm{cl}H)^\wedge)^{\mathbb{N}} : \ H \leq s_{\mathbf{u}}(\mathrm{cl}H)\}.$$

By the definition of $\mathfrak{g}$-closed subgroups, $H = s_S(\mathrm{cl}H)$. Since H is dense in $\mathrm{cl}H$, $S \in \mathcal{TBS}((\mathrm{cl}H)^\wedge)$ by Corollary 2.3. Now the assertion follows from Theorem 1.7(1). □

Let G be a MAP Abelian topological group, $X = \widehat{G}$ and α be the natural including of G into $G^{\wedge\wedge}$. Since G is MAP, α is injective. The weak and weak* group topologies on X we denote by τ_w and τ_{w^*} respectively, i.e., $\tau_w = \sigma(X, G)$ and $\tau_{w^*} = \sigma(X, G^{\wedge\wedge})$. The compact-open topology on X is denoted by τ_{co}. Then $\tau_w \subseteq \tau_{w^*} \subseteq \tau_{co}$. Let $t : X_d \to (X, \tau_{co})(= G^\wedge), t(x) = x$, be the natural continuous isomorphism and $\mathfrak{b} := t^\wedge$ be its conjugate continuous monomorphism. Set $bG := X_d^\wedge$. It is well known that the group bG with the continuous monomorphism $\mathfrak{b} \circ \alpha$ is the Bohr compactification of G (although α need not be continuous, but $\mathfrak{b} \circ \alpha$ is always continuous since $(\mathfrak{b} \circ \alpha(g), x) = (\alpha(g), t(x)) = (x, g)$ for every $g \in G$ and $x \in X_d$). We shall algebraically

identify G and $G^{\wedge\wedge}$ with their images $\mathfrak{b}\circ\alpha(G)$ and $\mathfrak{b}\,(G^{\wedge\wedge})$ respectively saying that they are subgroups of bG. It is clear that

$$\mathfrak{g}_{bG}(\mathfrak{b}\circ\alpha(G)) \subseteq \mathfrak{g}_{bG}(\mathfrak{b}\,(G^{\wedge\wedge}))\,. \tag{2.1}$$

Proposition 2.7. *Let G be a MAP Abelian topological group and $X = \widehat{G}$. The following statements are equivalent:*

(i) $\mathbf{s}(X, \tau_w) = \mathbf{s}(X, \tau_{w^*})$;

(ii) $\mathfrak{g}_{bG}(\mathfrak{b}\circ\alpha(G)) = \mathfrak{g}_{bG}(\mathfrak{b}\,(G^{\wedge\wedge}))$.

In particular, if G is reflexive, then (i) and (ii) are fulfilled.

Proof. (i)$\Rightarrow$(ii). By (2.1), we have to show that $\mathfrak{g}_{bG}(\mathfrak{b}\circ\alpha(G)) \supseteq \mathfrak{g}_{bG}(\mathfrak{b}\,(G^{\wedge\wedge}))$. Let $\mathbf{u} = \{u_n\}_{n\in\omega} \subset X$ be such that $\mathfrak{b}\circ\alpha(G) \subseteq s_{\mathbf{u}}(bG)$. This means that $(\mathfrak{b}\circ\alpha(g), u_n) = (u_n, g) \to 1$ for every $g\in G$, i.e., $\mathbf{u}\in S(X,\tau_w)$. By hypothesis, $\mathbf{u}\in S(X,\tau_{w^*})$ either. Hence

$$(\mathfrak{b}(\chi), u_n) = (\chi, u_n) \to 1 \text{ for every } \chi\in G^{\wedge\wedge},$$

i.e., $\mathfrak{b}(G^{\wedge\wedge}) \subseteq s_{\mathbf{u}}(bG)$. So $\mathfrak{g}_{bG}(\mathfrak{b}\circ\alpha(G)) \supseteq \mathfrak{g}_{bG}(\mathfrak{b}\,(G^{\wedge\wedge}))$.

(ii)$\Rightarrow$(i). Since $\tau_w \subseteq \tau_{w^*}$, we have to show only that if $\mathbf{u} = \{u_n\}_{n\in\omega} \in S(X,\tau_w)$, then also $\mathbf{u}\in S(X,\tau_{w^*})$. Assuming the converse we can find $\chi\in G^{\wedge\wedge}$ such that

$$(\chi, u_n) \not\to 1, \text{ at } n\to\infty.$$

Then $\mathfrak{b}(\chi)\not\in s_{\mathbf{u}}(bG)$. Thus $\mathfrak{b}(\chi)\not\in \mathfrak{g}_{bG}(\mathfrak{b}\circ\alpha(G))$. A contradiction.

Proof of Proposition 1.14. Since G is reflexive, $\tau_w = \tau_{w^*}$. By Proposition 2.7, we have $\mathfrak{g}_{bG}(\mathfrak{b}\circ\alpha(G)) = \mathfrak{g}_{bG}(\mathfrak{b}\,(G^{\wedge\wedge}))$. By [4, Theorem 2.4], the dual group of a separable metrizable Abelian group G is sequential. By [12, Theorem 1.13], $G^{\wedge}$ is a s-group. Hence, by Theorem 1.10, $\mathfrak{g}_{bG}(\mathfrak{b}\circ\alpha(G)) = \mathfrak{b}\circ\alpha(G)$ and $\mathfrak{b}\circ\alpha(G)$ is a $\mathfrak{g}$-closed subgroup of bG. $\Box$

Now we discuss the minimality of $|S|$ of TS-sets S which generate the same topology.

Definition 2.8. *Let G be an Abelian group.*

(1) *If $S \in \mathcal{TS}(G)$, we put*

$$r_s(S) = \inf\{|B| :\ (G,\tau_B) \cong (G,\tau_S) \ \textit{and}\ B \in \mathcal{TS}(G)\},$$

$$r_s^\wedge(S) = \inf\{|B| :\ s_B(G_d^\wedge) = s_S(G_d^\wedge) \ \textit{and}\ B \in \mathcal{TS}(G)\}.$$

(2) *If $S \in \mathcal{TBS}(G)$, we put*

$$r_b(S) = \inf\{|B| :\ (G,\tau_{bB}) \cong (G,\tau_{bS}) \ \textit{and}\ B \in \mathcal{TBS}(G)\},$$

$$r_b^\wedge(S) = \inf\{|B| :\ s_B(G_d^\wedge) = s_S(G_d^\wedge) \ \textit{and}\ B \in \mathcal{TBS}(G)\}.$$

Remark 2. Let (G,τ) be a s-group and $\tau = \tau_S$ for some $S \in \mathcal{TS}(G)$. Then the number $r_s(S)$ coincides with the number $r_s(G,\tau)$ that is defined in [12].

Proposition 2.9. *Let G be an infinite Abelian group.*

1) *If $S \in \mathcal{TS}(G)$, then $r_s^\wedge(S) \le r_s(S)$.*
2) *If $S \in \mathcal{TBS}(G)$, then $r_s^\wedge(S) = r_b^\wedge(S) = r_b(S)$.*
3) *If $S \in \mathcal{TS}(G)$ is finite, then $r_s(S) = r_s^\wedge(S) = 1$.*

Proof. 1) Let $B \in \mathcal{TS}(G)$ be such that $(G,\tau_B) \cong (G,\tau_S)$. By Theorem 1.7(1), algebraically,

$$s_B(G_d^\wedge) = (G,\tau_B)^\wedge = (G,\tau_S)^\wedge = s_S(G_d^\wedge).$$

So $|B| \ge r_s^\wedge(S)$. Thus $r_s^\wedge(S) \le r_s(S)$.

2) Let $S \in \mathcal{TBS}(G)$. By Corollary 2.3, $s_S(G_d^\wedge)$ is dense in $G_d^\wedge$. Hence, if $s_B(G_d^\wedge) = s_S(G_d^\wedge)$ for $B \in \mathcal{TS}(G)$, then, by Corollary 2.3, $B \in \mathcal{TBS}(G)$. Thus, $r_s^\wedge(S) = r_b^\wedge(S)$.

Let $B \in \mathcal{TBS}(G)$. By Theorem 1.8 and [5, Theorem 1.3], $s_B(G_d^\wedge) = s_S(G_d^\wedge)$ if and only if $\tau_{bB} = \tau_{bS}$. So $r_b(S) = r_b^\wedge(S)$.

3) By Proposition 2.6 of [12], $r_s(S) = 1$ and the assertion follows from item 1).

Example 2.1. Let (G,τ) be a dense countably infinite subgroup of a compact infinite metrizable Abelian group with the induced topology. Thus (G,τ) is a s-group. Set $S = S(G,\tau)$. Since $(G,\mathbf{u})$ is either discrete or non metrizable, by Proposition 2.9(3), we have $r_s(S) \ge \aleph_0$. On the other hand, by Theorem 1.7, algebraically, $s_S(G_d^\wedge) = (G,\tau)^\wedge$ is a countable subgroup of $G_d^\wedge$. So, by [6], there exists a TB-sequence $\mathbf{u}$ in G such that $s_{\mathbf{u}}(G_d^\wedge) = s_S(G_d^\wedge)$. Thus $r_s^\wedge(S) = 1$ and hence $r_s(S) > r_s^\wedge(S)$. We do not know any characterization of those s-groups for which $r_s(S) = r_s^\wedge(S)$. $\Box$

3. Structure of Abelian s-groups

In the following proposition we describe all sequences converging to zero in $(G, \mathbf{u})$.

Proposition 3.1. *Let $\mathbf{u} = \{u_n\}$ be a T-sequence in an Abelian group G. A sequence $\mathbf{v} = \{v_n\}$ converges to zero in $(G, \mathbf{u})$ if and only if there are $m \geq 0$ and $n_0 \geq 0$ such that for every $n \geq n_0$ each member $v_n \neq 0$ can be represented in the form*

$$v_n = a_1^n u_{k_1^n} + \cdots + a_{l_n}^n u_{k_{l_n}^n},$$

where $k_1^n < \cdots < k_{l_n}^n$, $|a_1^n| + \cdots + |a_{l_n}^n| \leq m+1$ and $k_1^n \to \infty$.

Proof. If either $\mathbf{u}$ or $\mathbf{v}$ is trivial, the proposition is evident. Assume that $\mathbf{u}$ and $\mathbf{v}$ are non-trivial. The sufficiency is clear. Let us prove the necessity. Since the subgroup $\langle \mathbf{u} \rangle$ of G is open in $\tau_{\mathbf{u}}$, there is n_0 such that $v_n \in \langle \mathbf{u} \rangle$ for every $n \geq n_0$. Thus, without loss of generality, we may assume that $\langle \mathbf{u} \rangle = G$. Since $\mathbf{v} \cup \{0\}$ is compact and $\langle \mathbf{u} \rangle = G$, by [12, Theorem 2.10], there is $m \geq 0$ such that $\mathbf{v} \subset A(m, 0)$. So, if $v_n \neq 0$, then

$$v_n = a_1^n u_{k_1^n} + \cdots + a_{l_n}^n u_{k_{l_n}^n}, \text{ where } k_1^n < \cdots < k_{l_n}^n \text{ and } |a_1^n| + \cdots + |a_{l_n}^n| \leq m+1. \tag{3.1}$$

Also we may assume that for any fix n every sum of terms of the form $a_i^n u_{k_i^n}$ in (3.1) is not equal to zero (in particular, $a_i^n u_{k_i^n} \neq 0$ for $i = 1, \dots, l_n$).

Now we have to show that $k_1^n \to \infty$. Assuming the converse and passing to a subsequence we may suppose that $k_1^n = k_1$, $a_1^n = a_1$ and $a_{k_1^n}^n u_{k_1^n} = a_1 u_{k_1} \neq 0$ for every n. So

$$v_n = a_1 u_{k_1} + a_2^n u_{k_2^n} + \cdots + a_{l_n}^n u_{k_{l_n}^n} = a_1 u_{k_1} + w_n^1.$$

If $k_2^n \to \infty$, we obtain that w_n^1 converges to zero. Hence $0 \neq a_1 u_{k_1} = v_n - w_n^1 \to 0$. This is a contradiction. Thus, there is a bounded subsequence of $\{k_2^n\}$. Passing to a subsequence we may suppose that $k_2^n = k_2$, $a_2^n = a_2$ and $a_2^n u_{k_2^n} = a_2 u_{k_2} \neq 0$ for every n. So

$$v_n = a_1 u_{k_1} + a_2 u_{k_2} + a_3^n u_{k_3^n} + \cdots + a_{l_n}^n u_{k_{l_n}^n} = a_1 u_{k_1} + a_2 u_{k_2} + w_n^2.$$

By hypothesis, $a_1 u_{k_1} + a_2 u_{k_2} \neq 0$. And so on. Since

$$0 < |a_1| < |a_1| + |a_2| < \cdots \leq m+1,$$

after at most $m+1$ steps, we obtain that there is a *fix* and *non-zero* subsequence of $\mathbf{v}$. Thus $v_n \not\to 0$. This contradiction shows that $k_1^n \to \infty$.

The following theorem makes more precise Theorem 2.9 of [12].

Theorem 3.2. *Let* $\mathbf{u} = \{u_n\}$ *be a* T*-sequence in an Abelian group* G *such that* $\langle \mathbf{u} \rangle = G$. *Then* $(G, \mathbf{u})$ *is a quotient group of* $(\mathbb{Z}_0^{\mathbb{N}}, \mathbf{e})$ *under the sequence-covering homomorphism*

$$\pi((n_1, n_2, \dots, n_m, 0, \dots)) = n_1 u_1 + n_2 u_2 + \cdots + n_m u_m.$$

Proof. Taking into account Theorem 2.9 of [12], we have to show only that π is sequence-covering. Let $\mathbf{v} = \{v_n\} \in S(G, \mathbf{u})$. By Proposition 3.1, for some natural number m we can represent every $v_n \neq 0$ in the form

$$v_n = a_1^n u_{k_1^n} + \cdots + a_{l_n}^n u_{k_{l_n}^n},$$

where $k_1^n < \cdots < k_{l_n}^n$, $|a_1^n| + \cdots + |a_{k_{l_n}^n}^n| \leq m + 1$ and $k_1^n \to \infty$. Set

$$e_n' = a_1^n e_{k_1^n} + \cdots + a_{l_n}^n e_{k_{l_n}^n} \text{ if } v_n \neq 0, \text{ and } e_n' = 0 \text{ if } v_n = 0.$$

Then $e_n' \to 0$ in $(\mathbb{Z}_0^{\mathbb{N}}, \mathbf{e})$ and $\pi(e_n') = v_n$.

Let $\{(G_i, \tau_i)\}_{i \in I}$, where I is a non-empty set of indices, be a family of Hausdorff topological groups. For every $i \in I$ fix $U_i \in \mathcal{U}_{G_i}$ and put

$$\sum_{i \in I} U_i := \left\{ (g_i)_{i \in I} \in \sum_{i \in I} G_i : \ g_i \in U_i \text{ for all } i \in I \right\}.$$

Then the sets of the form $\sum_{i \in I} U_i$, where $U_i \in \mathcal{U}_{G_i}$ for every $i \in I$, form a neighborhood basis at the unit of a Hausdorff group topology τ^r on $\sum_{i \in I} G_i$ that is called the rectangular (or box) topology.

Let $\mathbf{u} = \{g_n\}$ be an arbitrary sequence in $S(G_i, \tau_i)$. Evidently, the sequence $j_i(\mathbf{u})$ converges to zero in τ^r. Thus, the set $\bigcup_{i \in I} j_i\left(S(G_i, \tau_i)\right)$ is a TS-set of sequences in $\sum_{i \in I} G_i$. So, if (G_i, τ_i) is a s-group for all $i \in I$, then Definition 1.15 is correct. Moreover, we can prove the following:

Proposition 3.3. *Let* $G = \sum_{i \in I} G_i$, *where* (G_i, τ_i) *is a* s*-group for every* $i \in I$. *Set* $S := \bigcup_{i \in I} j_i\left(S(G_i, \tau_i)\right)$. *The topology* τ_S *on* G *coincides with the finest Hausdorff group topology* τ' *on* G *for which all inclusions* j_i *are continuous.*

PROOF. Fix $i \in I$. By construction, for every $\{u_n\} \in S(G_i, \tau_i)$, $j_i(u_n) \to e_G$ in τ_S. By [12, Theorem 2.4], the inclusion j_i is continuous. Thus, $\tau_S \subseteq \tau'$. Conversely, if j_i is continuous, then $j_i\left(S(G_i, \tau_i)\right) \subset S(G, \tau')$. Hence $S \subseteq S(G, \tau')$ and $\tau' \subseteq \tau_S$ by the definition of τ_S.

Theorem 3.4. *Let (X, τ) be an Abelian s-group. Set $I = S(X, \tau)$. For every $\mathbf{u} \in I$, let $p_{\mathbf{u}}(\langle \mathbf{u} \rangle, \mathbf{u}) \to X, p_{\mathbf{u}}(g) = g$, be the natural including of $(\langle \mathbf{u} \rangle, \mathbf{u})$ into X. Then the natural homomorphism*

$$p : s - \sum\nolimits_{\mathbf{u} \in S(X,\tau)} (\langle \mathbf{u} \rangle, \mathbf{u}) \to X, \; p\left((x_{\mathbf{u}})\right) = \sum_{\mathbf{u}} p_{\mathbf{u}}(x_{\mathbf{u}}) = \sum_{\mathbf{u}} x_{\mathbf{u}},$$

is a quotient sequence-covering map.

PROOF. Set

$$G := s - \sum\nolimits_{\mathbf{u} \in S(X,\tau)} (\langle \mathbf{u} \rangle, \mathbf{u}) \text{ and } S := \bigcup_{\mathbf{u} \in S(X,\tau)} j_{\mathbf{u}}(S(\langle \mathbf{u} \rangle, \mathbf{u})) \in \mathcal{TS}(G).$$

Since any element of X can be regarded as the first element of some sequence $\mathbf{u} \in S(X, \tau)$, p is surjective. By construction, p is sequence-covering.

Let $\mathbf{v} = \{v_n\} \in S$. By construction, $p(v_n) = v_n \to 0$ in τ. Thus, by [12, Theorem 2.4], p is continuous. Set $H = \ker p$. By [12, Theorem 1.11], $G/H \cong (X, \tau_{p(S)})$. Since, by construction, $p(S) = S(X, \tau)$, we obtain that $G/H \cong (X, \tau)$ by Proposition 1.7 of [12].

To prove Theorem 1.16 we need the following proposition.

Proposition 3.5. *Let $\{(X_i, \nu_i)\}_{i \in I}$ and $\{(G_i, \tau_i)\}_{i \in I}$ be non-empty families of Abelian s-groups and let $\pi_i : G_i \to X_i$ be a quotient sequence-covering map for every $i \in I$. Set $X = s - \sum_{i \in I} X_i$, $G = s - \sum_{i \in I} G_i$ and $\pi : G \to X, \pi((g_i)) = (\pi_i(g_i))$. Then π is a quotient map.*

PROOF. It is clear that π is surjective. Set

$$S_X := \bigcup_{i \in I} j_i(S(X_i, \nu_i)) \text{ and } S_G := \bigcup_{i \in I} j_i(S(G_i, \tau_i)).$$

Since π_i is sequence-covering, we have $\pi_i(S(G_i, \tau_i)) = S(X_i, \nu_i)$. Hence $\pi(S_G) = S_X$. Thus, by [12, Theorem 2.4], π is continuous. By [12, Theorem 1.11], $G/\ker(\pi) \cong (X, \tau_{\pi(S_G)})$. Hence $G/\ker(\pi) \cong X$ and π is a quotient map.

Proof of Theorem 1.16. Let $I = S(X,\tau)$. For every $\mathbf{u} \in I$, put $G_{\mathbf{u}} = (\mathbb{Z}_0^{\mathbb{N}}, \mathbf{e})$, $X_{\mathbf{u}} = (\langle \mathbf{u}\rangle, \mathbf{u})$ and $\pi_{\mathbf{u}}((n_1,\dots,n_m,0,\dots)) = n_1u_1 + \cdots + n_m u_m$. Let $p_{\mathbf{u}}(\langle \mathbf{u}\rangle, \mathbf{u}) \to X, p_{\mathbf{u}}(g) = g$, be the natural including of $(\langle \mathbf{u}\rangle, \mathbf{u})$ into X. Then the theorem immediately follows from Theorems 3.2 and 3.4 and Proposition 3.5. $\Box$

The following theorem is a natural counterpart of [18, Theorem 4.1]:

Theorem 3.6. *Let (X,τ) be a non-trivial Hausdorff Abelian topological group. The following statements are equivalent:*

(i) *(X,τ) is a s-group;*
(ii) *every continuous sequence-covering homomorphism from an Abelian s-group onto (X,τ) is quotient.*

Proof. (i) $\Rightarrow$ (ii). Let $p : G \to X$ be a sequence-covering continuous homomorphism from a s-group (G,ν) onto X. Set $H = \ker p$. We have to show that p is quotient, i.e., $X \cong G/H$. Since p is surjective, by [12, Theorem 1.11], we have $G/H \cong (X, \tau_{p(S(G,\nu))})$. By hypothesis and Proposition 1.7 of [12], $p(S(G,\nu)) = S(X,\tau)$ and $\tau = \tau_{S(X,\tau)}$. Thus $G/H \cong X$.

(ii) $\Rightarrow$ (i). Let $I = S(X,\tau)$, $S := \bigcup_{\mathbf{u}\in S(X,\tau)} j_{\mathbf{u}}(S(\langle \mathbf{u}\rangle, \mathbf{u})) \in \mathcal{TS}(G)$, $G := s - \sum_{\mathbf{u}\in S(X,\tau)} (\langle \mathbf{u}\rangle, \mathbf{u})$ and

$$p : G \to X,\ p\left((x_{\mathbf{u}})\right) = \sum_{\mathbf{u}} p_{\mathbf{u}}(x_{\mathbf{u}}) = \sum_{\mathbf{u}} x_{\mathbf{u}},$$

Since every $(\langle \mathbf{u}\rangle, \mathbf{u})$ is a s-group, G is a s-group either. By [12, Theorem 2.4], p is continuous. Since p is sequence-covering, by hypothesis, p is quotient. Thus $(X,\tau) \cong G/\ker p$. By Theorem [12, Theorem 1.11], we also have $G/\ker p \cong (X, \tau_{p(S)})$. Thus $\tau = \tau_{p(S)}$ and (X,τ) is a s-group.

4. Countable s-sums of s-groups

We start from the description of the topology τ_S on G for countably infinite $S \in \mathcal{TS}(G)$.

Proposition 4.1. *Let $S = \{\mathbf{u}_n\}_{n\in\omega} \in \mathcal{TS}(G)$. Then the family $\mathcal{U}$ of all the sets of the form*

$$\sum_n W_n = \bigcup_{n=0}^{\infty} (W_0 + W_1 + \cdots + W_n),\ \textit{where } 0 \in W_n \in \tau_{\mathbf{u}_n},\ n \geq 0,$$

forms an open basis at 0 of τ_S.

Proposition 4.1 is an immediate corollary of the following two assertions.

Lemma 4.2. *Let $S \in \mathcal{TS}(G)$ for an Abelian group G and $S = \cup_{i\in I} S_i$, where I is a non-empty set of indices. Then $\tau_S \subseteq \bigwedge_i \tau_{S_i}$.*

Proof. It is clear that $S_i \in \mathcal{TS}(G)$ and $\tau_S \subseteq \tau_{S_i}$ for every $i \in I$. Thus, if $U \in \tau_S$, then $U \in \tau_{S_i}$ for every $i \in I$. Hence $\tau_S \subseteq \bigwedge_i \tau_{S_i}$.

Proposition 4.3. *Let $S = \cup_{n=0}^{\infty} S_n \in \mathcal{TS}(G)$. Then the family $\mathcal{U}$ of all the sets of the form*

$$\sum_n W_n := \bigcup_{n=0}^{\infty} (W_0 + W_1 + \dots + W_n), \text{ where } 0 \in W_n \in \tau_{S_n},\ n \geq 1,$$

is an open basis at 0 of τ_S.

Proof. It is clear that $S_n \in \mathcal{TS}(G)$ for every n.

1. We claim that *$\mathcal{U}$ forms an open basis at zero of a Hausdorff group topology τ on G*. For this we have to check the five conditions of Theorem 4.5 of [14].

Let $\sum_n W_n \in \mathcal{U}$. To prove (i) choose $V_n \in \tau_{S_n}$ such that $V_n + V_n \subseteq W_n$. Then

$$\sum_n V_n + \sum_n V_n \subseteq \sum_n (V_n + V_n) \subseteq \sum_n W_n.$$

(ii) and (iv) are trivial.

To prove (iii) let $g = w_{n_1} + \dots + w_{n_m} \in \sum_n W_n$, where $w_{n_k} \neq 0, k = 1, \dots, m$. If $n \notin \{n_1, \dots, n_m\}$, we set $V_n = W_n$. If $n = n_k$, we may choose an open neighborhood V_{n_k} of zero in $\tau_{S_{n_k}}$ such that $w_{n_k} + V_{n_k} \subset W_{n_k}$. Then $g + \sum_n V_n \subseteq \sum_n W_n$.

To prove (v) let $\sum_n V_n, \sum_n W_n \in \mathcal{U}$. Set $F_n = W_n \cap V_n \in \tau_{S_n}$. Then $\sum_n F_n \subseteq \sum_n V_n \cap \sum_n W_n$.

2. We claim that $\tau \subseteq \tau_S$. By the definition of τ_S we have to show only that every $\mathbf{u} = \{u_k\} \in S_n$ converges to zero in τ. Let $\sum_n W_n \in \mathcal{U}$. Since $W_n \in \tau_{S_n}$, $u_k \in W_n \subset \sum_n W_n$ for all sufficiently large k. Thus $\mathbf{u}$ converges to zero in τ.

3. We claim that $\tau = \tau_S$. Let $U \in \tau_S$ be an arbitrary neighborhood of zero. Then there is a sequence of open neighborhoods of zero $U_n \in \tau_S, n \geq 0$, such that $U_0 + U_0 \subseteq U$ and $U_n + U_n \subseteq U_{n-1}, n \geq 1$. By Lemma 4.2, $\tau_S \subseteq \bigwedge_n \tau_{S_n}$. Hence for every $n \geq 0$ we may choose an open neighborhood W_n of zero in τ_{S_n} such that $W_n \subset U_n$. It is clear that $\sum_n W_n \subseteq U$.

To prove Theorem 1.17, we need the following proposition.

Proposition 4.4. *Let $\{G_n\}_{n\in\omega}$ be a sequence of Abelian groups and let $\mathbf{u}_n$ be a T-sequence in G_n for every $n\in\omega$. Set $G=\sum_{n\in\omega} G_n$ and $S=\{j_n(\mathbf{u}_n)\}_{n\in\omega}$. Then (G,τ_S) is a complete sequential group, $\tau_S=\tau^r$ and*

$$(G,\tau_S)^\wedge = \prod_{n\in\omega} (G_n,\mathbf{u}_n)^\wedge .$$

Moreover, if all G_n are countably infinite, then $(G,\tau_S)^\wedge$ is a Polish group.

Proof. (G,τ_S) is a complete sequential group by Theorem 2.7 of [12]. By Proposition 4.1, $\tau_S=\tau^r$. Thus, by [15], $(G,\tau_S)^\wedge=\prod_{n\in\omega}(G_n,\mathbf{u}_n)^\wedge$. If all G_n are countably infinite, then, by Theorem 1.2, all $(G_n,\mathbf{u}_n)^\wedge$ are Polish. Hence $(G,\tau_S)^\wedge$ is a Polish group either.

Proof of Theorem 1.17. Set $G'=\sum_{n\in\omega} G_n$, where $G_n=G$ for every $n\in\omega$, and $S'=\{j_n(\mathbf{u}_n)\}_{n\in\omega}$. Then, by Proposition 4.4,

$$(G',\tau_{S'})^\wedge = \prod_{n\in\omega} (G,\mathbf{u}_n)^\wedge$$

is a Polish group.

Set $p:(G',\tau_{S'})\to (G,\tau_S), p((g_n))=\sum_n g_n$. Since $p(j_n(\mathbf{u}_n))=\mathbf{u}_n$ converges to zero in (G,τ_S), p is continuous by Theorem 2.4 of [12]. Set $H=\ker p$. Since $p(S')=S$, by [12, Theorem 1.11], $(G,\tau_S)\cong (G',\tau_{S'})/H$. Then the conjugate homomorphism $p^\wedge$ is a continuous isomorphism from $(G,\tau_S)^\wedge$ onto the annihilator $H^\perp$ of H in $(G',\tau_{S'})^\wedge$. By [12, Theorem 2.7], every compact subset of (G,τ_S) is contained in a compact subset K_n of the form

$$K_n := \left[\bigcup_{i=0}^{n} (\mathbf{u}_i\cup(-\mathbf{u}_i))\right] + \dots + \left[\bigcup_{i=0}^{n} (\mathbf{u}_i\cup(-\mathbf{u}_i))\right]$$

with $n+1$ summands. It is clear that a subset K'_n of G' of the form

$$K'_n := \left[\bigcup_{i=0}^{n} (j_i(\mathbf{u}_i)\cup(-j_i(\mathbf{u}_i)))\right] + \dots + \left[\bigcup_{i=0}^{n} (j_i(\mathbf{u}_i)\cup(-j_i(\mathbf{u}_i)))\right]$$

with $n+1$ summands, is compact. Since $p(K'_n)=K_n$ and p is onto and continuous, p is compact-covering. Thus, by [1, Lemma 5.17], $p^\wedge$ is an embedding of $(G,\tau_S)^\wedge$ into the Polish group $(G',\tau_{S'})^\wedge$. So $(G,\tau_S)^\wedge\cong H^\perp$ is a Polish group. $\Box$

Proof of Proposition 1.18. For every X_n there is a countably infinite Abelian group G_n and a TB-sequence $\mathbf{u}_n$ in G_n such that $(G_n, \mathbf{u}_n)^\wedge \cong X_n$ [11]. Set $G = \sum_{n\in\omega} G_n$ and $S = \{j_n(\mathbf{u}_n)\}_{n\in\omega}$. Then the proposition follows from Proposition 4.4. □

Proof of Proposition 1.19. By Proposition 2.9 of [11], there is a TB-sequence $\mathbf{u}$ on $\mathbb{Z}^2$, such that $(\mathbb{Z}^2, \mathbf{u})^\wedge \cong \mathbb{R}$. Since $\sum_{n\in\omega} \mathbb{Z}^2 \cong \mathbb{Z}_0^{\mathbb{N}}$, the assertion follows from Proposition 4.4. □

5. Open questions

We start from a question concerning Theorem 1.10:

Problem 5.1. *Let X be a compact Abelian group and H be a $\mathfrak{g}$-closed non-dense subgroup of X. Is there $S \in \mathcal{TS}(\widehat{X})$ such that $i_S^\wedge\left((\widehat{X}, \tau_S)^\wedge\right) = H$?*

As it was noted, if G is a *separable* metrizable Abelian topological group, then, by [4, Theorem 1.7], the dual group $G^\wedge$ is sequential. Hence $G^\wedge$ is a s-group. The following questions are open:

Problem 5.2. *Let G be a non-separable metrizable (resp. Fréchet-Urysohn or sequential) Abelian group. Is $G^\wedge$ a s-group?*

Problem 5.3. *Let G be an Abelian s-group. When $G^\wedge$ is a s-group?*

Problem 5.4. *Let G be an Abelian (resp. metrizable, Fréchet-Urysohn, sequential or a s-group) topological group such that $G^\wedge$ is a (resp. metrizable, Fréchet-Urysohn or sequential) s-group. What we can say additionally about G and $G^\wedge$?*

For example, if G is metrizable and $G^\wedge$ is Fréchet-Urysohn, then, by [4, Theorem 2.2], $G^\wedge$ is locally compact metrizable group.

Let G be an Abelian group and $S \in \mathcal{TBS}(G)$. Theorem 1.8 gives a complete description of the topology τ_{bS} on G. On the other hand, we do not know any description of the topology on the dual group.

Problem 5.5. *Describe the topology of $(G, \tau_{bS})^\wedge$.*

By Corollary 1.9, $(G, \tau_{bS})^\wedge = (G, \tau_S)^\wedge$ algebraically. It is natural to ask:

Problem 5.6. *When the groups $(G, \tau_{bS})^\wedge$ and $(G, \tau_S)^\wedge$ are topologically isomorphic? In particular, when $(G, \tau_{\mathbf{u}})^\wedge \cong (G, \tau_{b\mathbf{u}})^\wedge$?*

Let G be a countably infinite Abelian group and $S \in \mathcal{TBS}(G)$. By Corollary 2.5, if $(G, \tau_{bS})^\wedge$ is countable, then (G, τ_{bS}) is not reflexive.

Problem 5.7. *Is there a $S \in \mathcal{TBS}(G)$ for a* countably *infinite Abelian group G such that (G, τ_{bS}) is reflexive?*

Note that the positive answer to the last question will give the positive answer to the following general problem:

Problem 5.8. (M. G. Tkachenko) *Is there a reflexive precompact group topology on a* countably *infinite Abelian group (for example, on $\mathbb{Z}$)?*

Taking into consideration of Corollary 1.13 and Proposition 1.14, one can ask:

Problem 5.9. *Which MAP Abelian groups are $\mathfrak{g}$-closed in its Bohr compactification?*

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